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Letter to the Editor

A Better Difference Scheme for the Laplace Equation in Cylindrical Coordinates

1. Introduction

We shall present a highly accurate 9-point difference scheme for the Laplace equation $\Delta u(r, z) = 0$ in cylindrical coordinates. By using Fourier techniques, we shall show that the computed solution for this scheme is more accurate by at least a factor of ten than that for other 9-point schemes.

In an earlier paper [1], we analyzed several difference equations which approximate the solution to the Laplace equation $\partial^2 u/\partial r^2 + (1/r)\,\partial u/\partial r + \partial^2 u/\partial z^2 = 0$. There, the difference approximation was Fourier decomposed in the z-direction and the behaviour of the Fourier components was compared to the exact solution. For small values of the frequency ω, the difference was presented as a function of r and ω. A more detailed description is given in [1]. Using this method we were able to find which of the difference schemes under consideration gave the most accurate solution to the difference equations. We shall now apply the same technique to two other schemes, one 9-point scheme [2] and one 5-point scheme [3]. They give solutions which have errors $O(h^4)$ and $O(h^2)$, respectively.

2. Nine-Point Scheme

Let $v(i, j)$ denote the difference approximation to $u(ih, jh) = u(r, z)$. Then the 9-point scheme for interior (off-axis) points from [2] reads

$$\begin{aligned}(20 - 14/(20i^2 - 5))\,v(i, j) = {} & (1 + 1/(2i) - 3/(40i^2 + 20i))(v(i + 1, j - 1) \\ & + v(i + 1, j + 1)) + (1 - 1/(2i) - 3/(40i^2 - 20i)) \\ & \times (v(i - 1, j - 1) + v(i - 1, j + 1)) \\ & + (4 + 2/i - 7/(20i^2 + 10i))\,v(i + 1, j) \\ & + (4 - 2/i - 7/(20i^2 - 10i))\,v(i - 1, j) \\ & + (4 + 3/(20i^2 - 5))(v(i, j - 1) + v(i, j + 1)),\end{aligned}$$

where $1 \leqslant i \leqslant N - 1$, $Nh = 1$. The value $h = \frac{1}{20}$ was used in Figs. 1 and 2.

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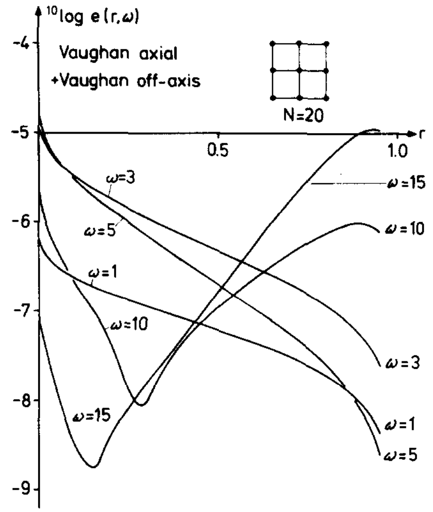

FIG. 1. The logarithm of the error $\hat{e}(r, \omega)$ for the 9-point formula used in [2]. Note that the error increases markedly for small values of r. This is because the approximation used for $r = 0$ is only $O(h^2)$, while the off-axis formula is $O(h^4)$.

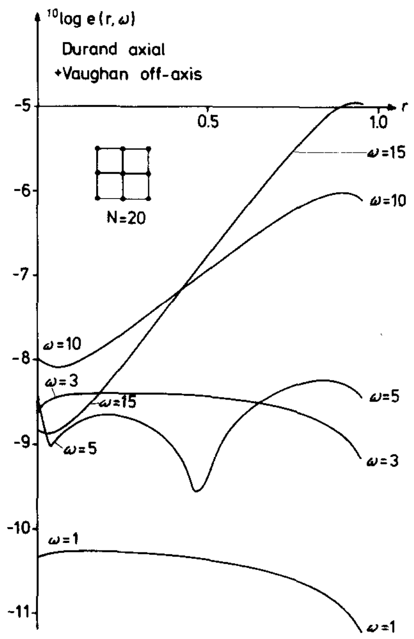

FIG. 2. The logarithm of the error $\hat{e}(r, \omega)$ for the most accurate 9-point difference scheme, for the Laplace equation in cylindrical coordinates.

We have combined the off-axis scheme above with several different approximations on the z-axis (the axis of symmetry). The axial extrapolation scheme suggested in [2] was found to give a very marked increase in the error close to the z-axis. See Fig. 1. The most accurate overall solutions were found using Durand's axial formula [4]: $v(0, j) = (5v(0, j-1) + 5v(0, j+1) + 7v(1, j-1) + 34v(1, j) + 7v(1, j+1))/58$. The error $\hat{e}(r, \omega)$ for this scheme is shown in Fig. 2 for different frequencies. Note that for a given problem, these curves must be weighted according to the Fourier decomposition of the boundary values. Comparison with [1, Fig. 7] (Durand axial + Durand off-axis) shows that the solutions here is more accurate by at least a factor of ten.

3. Five-Point Scheme

The 5-point formula used in [3] was also analyzed using the method described in [1]. The formula is

$$\begin{aligned} 2i(8i^2 - 5)\, v(i, j) = {} & (4i^3 + 2i^2 - 4i + 1)\, v(i+1, j) \\ & + (4i^3 - 2i^2 - 4i - 1)\, v(i-1, j) \\ & + i(4i^2 - 1)\, v(i, j+1) + i(4i^2 - 1)\, v(i, j-1), \end{aligned}$$

and

$$6 \cdot v(0, j) = 4v(1, j) + v(0, j+1) + v(0, j-1),$$

using the same notations as above.

From curves for the 5-point scheme, analogous to those in Figs. 1 and 2 above, we see that this scheme is better by a factor of two than the usual 5-point scheme, at least for low frequencies. Several different axial schemes (including Durand's, which is $O(h^4)$) were tried, but all gave the same result as long as they were at least $O(h^2)$.

4. Discussion

The difference schemes stated above are the most accurate ones known at present. There may very well exist other 9-point or 5-point schemes that are even better, but the task of determining the *optimal* coefficients seems overwhelming.

Note that a direct comparison of the leading terms in the truncation error is not possible, since this is composed of several derivatives of different orders, with different polynomials in r as coefficients. It is easier to use an explicitly known trial function, but the result then depends strongly on the choice of function and the chosen mesh-point. Both these methods, however, only give information on the local error of the difference schemes (i.e., when the neighbouring points are assumed to have exact values).

The Fourier technique used here and in [1] gives information on the global error of the difference schemes. It includes the effect of the approximation used at the axis of symmetry and at the outer boundaries. For smooth problems, it is sufficient to consider low frequencies, while for rapidly varying solutions, higher frequencies and smaller mesh-sizes h must be considered.

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